

Controlled options: derivatives with added flexibility

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Abstract

The paper introduces a limit version of multiple stopping options such that the holder selects dynamically a weight function that control the distribution of the payments (benefits) over time. In applications for commodities and energy trading, a control process can represent the quantity that can be purchased by a fixed price at current time. In another example, the control represents the weight of the integral in a modification of the Asian option. The pricing for these options requires to solve a stochastic control problem. Some existence results and pricing rules are obtained via modifications of parabolic Bellman equations.

Key words: stochastic control, exotic options, passport options, controlled options, multi-exercise options, continuous time market models, Bellman equation

JEL classification: G13, D81, C61

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1 Introduction

There are many different types of options and financial derivatives: European, American, Asian, Bermudian, Israeli, Russian, Parisian options, etc. (see, e.g., Briys *et al* (1998)). Pricing of exotic options require special methods; see, e.g., Kifer (2000), Kyprianou (2004), Kramkov *et al* (1994), Meinshausen and Hambly (2004), Peskir (2005), Bender and Schoenmakers (2006),

Bender (2011), Carmona and Dayanik (2006), Carmona and Touzi (2006), Dai and Kwok (2008), Dokuchaev (2009).

Typically, new type of options are developed with the purpose to offer some additional flexibility for an option holder. In particular, multiple exercise options are used in energy trading; allow a holder to distribute the purchase of energy by a fixed price over a set of time moments; see, e.g., Meinshausen and Hambly (2004), Bender and Schoenmakers (2006), Bender (2011), Carmona and Dayanik (2006), Carmona and Touzi (2006), Dai and Kwok (2008), Dokuchaev (2009), and Bender (2011). The paper suggests a next step in this direction. We develop a family of options that allows the holder to select dynamically continuous time processes that control the payoff. We call the new options *controlled options*. The control processes are assumed to be adapted to the current flow of information. More precisely, the holders of the new options select dynamically the weight functions that control the distribution of the payments (benefits) over time. There is a similarity with passport options introduced in Hyer *et al* (1997) as a generalization of the American option (see also Delbaen and Yor (2002), Kampen (2008), Nagayama (1999)). The passport options allow the holder to select investment strategies for an account; the writer guarantees protection from the losses. The difference with the control options introduced in this paper is that the holder of passport options selects portfolio strategy.

The controlled options may have applications in commodities and energy trading. For instance, control process $u(t)$ may represent the weight of the integral in a modification of the Asian option. In another example, a non-negative control process $u(t)$ can represent the amount of some commodity that can be purchased by a certain given price at time $t \in [0, T]$, where T is the terminal time, given that $\int_0^T u(t)dt = 1$. This is a limit case of multi-exercise option studied in Bender and Schoenmakers (2006) and Bender (2011), where the distribution of exercise times approaching a continuous distribution. Therefore, controlled options can be used also as an auxiliary tool to study these multi-exercise options. In some cases, analysis of these controlled option is more straightforward since optimal multi-stopping is actually excluded; it is replaced by more standard stochastic control problem. These and other examples of controlled options are studied below. It is shown that pricing for these options requires solution of a stochastic control problem rather than optimal stopping problem. Some existence results pricing rules are obtained in Markov diffusion setting based on dynamic programming and various modifications

of degenerate parabolic Bellman equations (Hamilton-Jacobi-Bellman (HJB) equation).

The paper is organized as follows. In Section 2, two classes of controlled options are introduced: (i) options where the adapted weight $u(t)$ is selected such that $\int_0^1 u(t)dt = 1$, and (ii) options where the weight $u(t)$ does not restriction on its cumulate, and where the payoff is defined by the normalized weight $v(t) = \left(\int_0^T u(s)ds\right)^{-1}$ which is not adapted. Some motivation for this setting is given. In Section 3, the market model is introduced. In Section 4, the general martingale pricing formula is given. In Section 5, the pricing is discussed for the case (i). In Section 6, the pricing is discussed for the case (ii). The proofs are given in Appendix.

2 Controlled options: definition and examples

Consider a risky asset (stock, commodity, a unit of energy) with the price $S(t)$, where $t \in [0, T]$, for a given $T > 0$. Consider an option with the payoff

$$F_u = \Phi(u(\cdot), S(\cdot)). \quad (2.1)$$

This payoff depends on a control process $u(\cdot)$ that is selected by an option holder from a certain class of admissible controls \mathcal{U} . The mapping $\Phi : \mathcal{U} \times \mathcal{S} \rightarrow \mathbf{R}$ is given; \mathcal{S} is the set of paths of $S(t)$. All processes from \mathcal{U} has to be adapted to the current information flow, i.e., adapted to some filtration \mathcal{F}_t that describes this information flow.

We call the corresponding options controlled options. Clearly, an American option is a special case of controlled options, where the exercise time is selected. Some new examples of controlled options are suggested and discussed below.

For simplicity, we assume that all options give the right on the corresponding payoff of the amount F_u in cash rather than the right to buy or sell stock or commodities.

Options with adapted weight with fixed cumulated integral

Consider a risky asset with the price $S(t)$. Let $T > 0$ be given, and let $g : \mathbf{R} \rightarrow \mathbf{R}$ and $f : \mathbf{R} \times [0, T] \rightarrow \mathbf{R}$ be some functions. Consider an option with the payoff at time T

$$F_u = g\left(\int_0^T u(t)f(S(t), t)dt\right), \quad (2.2)$$

Here $u(t)$ is the control process that is selected by the option holder. The process $u(t)$ has to be adapted to the filtration \mathcal{F}_t describing the information flow. In addition, it has to be selected such that

$$\int_0^T u(t)dt = 1.$$

A possible modification is the option with the payoff

$$F_u = F_u = \int_0^T u(t)f(S(t), t)dt + \left(1 - \int_0^T u(t)dt\right) f(S(T), T).$$

In this case, the unused $u(t)$ are accumulated and used at the terminal time.

Let us consider some examples of possible selection of f and g .

We denote $x^+ \triangleq \max(0, x)$.

Important special cases are the options with $g(x) = x$, $g(x) = (x - k)^+$, $g(x) = (K - x)^+$, $g(x) = \min(M, x)$, where $M > 0$ is the cap for benefits, and with

$$f(x, t) = x, \quad f(x, t) = (x - K)^+, \quad f(x, t) = (K - x)^+, \quad (2.3)$$

or

$$f(x, t) = e^{r(T-t)}(x - K)^+, \quad f(x, t) = e^{r(T-t)}(K - x)^+, \quad (2.4)$$

where $K > 0$ is given and where $r > 0$ is the risk-free rate.

Options (2.3) correspond to the case when the payments are made at current time $t \in [0, T]$, and options (2.4) correspond to the case when the payment is made at terminal time T . This model takes into account accumulation of interest up to time T on any payoff.

The option with payoff (2.2) with $f(x, t) \equiv x$ represents a generalization of Asian option where the weight $u(t)$ is selected by the holder.

The option with payoff (2.2) with $g(x) \equiv x$ represents a limit version of the multi-exercise options, when the distribution of exercise time approaches a continuous distribution. An additional restriction on $|u(t)| \leq \text{const}$ would represent the continuous analog of the requirement for multi-exercise options that exercise times must be on some distance from each other. For an analog of the model without this condition, strategies that may approach delta-functions.

These options can be used, for instance, for energy trading with $u(t)$ representing the quantity of energy purchased at time t for the fixed price K when the market price is above K . In this

case, the option represents a modification of the multi-exercise call option with continuously distributed payoff time. For this model, the total amount of energy that can be purchased is limited per option. Therefore, the option holder may prefer to postpone the purchase if she expects better opportunities in future.

2.1 Option with non-adapted normalized weight

A possible modification of the option described above is the option with the payoff at time T

$$F_u = g \left(\int_0^T v(t) f(S(t), t) dt \right), \quad (2.5)$$

where $g : \mathbf{R} \rightarrow \mathbf{R}$ and $f : \mathbf{R} \times [0, T] \rightarrow \mathbf{R}$ are given functions, the process $v(t)$ is such that

$$\int_0^T v(t) dt = 1.$$

The difference with option (2.2) is that the process $v(t)$ is not assumed to be adapted to the filtration \mathcal{F}_t generated by the current information flow. It is formed as

$$v(t) = \frac{u(t)}{\int_0^T u(s) ds}, \quad (2.6)$$

where the process $u(t)$ is selected by the option holder dynamically, using the current flow of information, i.e., it has to be adapted to the filtration \mathcal{F}_t describing this flow. The process $u(t)$ can be called weight process, and $v(t)$ can be called normalized wealth process.

This setting means that the option holder keeps the writer informed about her current selection of the value of $u(t)$, and these choices are recorded; the payoff occurs at terminal time T .

We don't exclude the case when $d_0 = 0$ and $u(t)|_{t \in [0, T)} = 0$. In this case, the payoff can be set by different ways. A possible way is to define the payoff for $u(t) \equiv 0$ as $F_u = g \left(\frac{1}{T} \int_0^T S(t) dt \right)$, i.e., as the limit of the payoff (2.5) for $u(t) \equiv \varepsilon$ as $\varepsilon \rightarrow 0$. Another possible selection of the payoff for $u(t) \equiv 0$ is $F_u = g(f(S(T), T))$, i.e., as the limit of the payoff (2.5) for $u(t) \equiv \varepsilon \mathbb{I}_{\{t \geq T-\varepsilon\}}$ as $\varepsilon \rightarrow 0$. In this case, $v(t)$ can be interpreted as the delta function with the mass concentrated at $t = T$.

These options can be useful generalizations of Asian options. Consider, for instance, a customer who consumes time variable and random quantity $u(t)$ of energy per time period

$(t, t + dt)$, with the price $S(t)$ for a unit. The cumulated number of units consumed up to time T is $\bar{u} = \int_0^T u(t)dt$; it is unknown at times $t < T$. To hedge against the price rise, the customer would purchase a portfolio of M call options; each option gives the right to purchase one energy unit for the price K . To minimize the impact of price fluctuations, the Asian options are commonly used. These options can be described as the options with the payoff $(\bar{S} - K)^+$, where $\bar{S} = T^{-1} \int_0^T S(t)dt$. For accounting and tax purposes, the average price of energy for a particular customer has to be calculated as $\bar{S}_u = \bar{u}^{-1} \int_0^T u(t)S(t)dt$ rather than \bar{S} . Therefore, more certainty in financial and tax situation can be achieved if one uses the portfolio of M options with payoff $(\bar{S}_u - K)^+$ that is defined by the consumption of the particular customer. This is a special case of option (2.5). Since \bar{u} is random and unknown, options (2.2) cannot be used for this model.

On impact of fixing the cumulated $u(t)$

It may appear that options with payoffs (2.5) are equivalent to the related options (2.2). However, the nature of control for these options is different.

First, the selection of $u(t)$ is obviously more restricted for options (2.2) than for options (2.5): the option holder have to obey the restrictions on the total amount of cumulated $u(t)$. Second, these two types of the options have different opportunities with respect to possibility to correct past decisions. Consider a model where the option holder selects $u(t)$ with the purpose to maximize the payoff F_u . For the holders of options (2.5), it is possible to smooth the effect of unfortunate decisions made at previous times by selecting larger $u(t)$ at future times. In addition, the relative weight of the past good decisions can be enlarged via selecting small current $u(t)$. This opportunity is absent for options (2.2).

3 Market model

We investigate pricing of the options described above for the simplest case of Black-Scholes model, i.e, for a complete continuous time diffusion market model with constant volatility. We consider the model of a securities market consisting of a risk free bond or bank account with the price $B(t)$ and a risky stock with the price $S(t)$, $t \in [0, T]$, where $T > 0$ be given terminal time.

Up to the end of this paper, we assume that the prices of the stocks evolves as

$$dS(t) = S(t) (a(t)dt + \sigma dw(t)), \quad (3.1)$$

where $a(t)$ is an appreciation rate, $\sigma > 0$ is a volatility coefficient.

In (3.1), $w(\cdot)$ is a standard Wiener process on a given standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\Omega = \{\omega\}$ is a set of elementary events, \mathcal{F} is a complete σ -algebra of events, and \mathbf{P} is a probability measure.

The price of the bond evolves as

$$B(t) = e^{rt}B(0). \quad (3.2)$$

We assume that $\sigma > 0$, $r \geq 0$, $B(0) > 0$, and $S(0) > 0$, are given constants.

Let \mathcal{F}_t be the filtration generated by $w(t)$. For simplicity, we assume that $a(t)$ is a bounded process progressively measurable with respect to \mathcal{F}_t . In this case, \mathcal{F}_t is also the filtration generated by $S(t)$.

Let \mathbf{P}_* be the probability measure such that the process $e^{-rt}S(t)$ is a martingale under \mathbf{P}_* on $[0, T]$. By the assumptions for (a, σ, r) , this measure exists and it is unique. Let \mathbf{E}_* be the corresponding expectation. Under the risk neutral measure \mathbf{P}_* ,

$$w_*(t) \triangleq w(t) + \int_0^t \sigma^{-1}[a(s) - r]ds$$

is a Wiener process, and the process $\tilde{S}(t)$ is a martingale, since $a(t)dt + \sigma dw(t) = \sigma dw_*(t)$ and $d\tilde{S}(t) = \sigma dw_*(t)$.

Admissible portfolio strategies

Let $X(0) > 0$ be the initial wealth at time $t = 0$ and let $X(t)$ be the wealth at time $t > 0$. We assume that the wealth $X(t)$ at time $t \in [0, T]$ is

$$X(t) = \beta(t)B(t) + \gamma(t)S(t). \quad (3.3)$$

Here $\beta(t)$ is the quantity of the bond portfolio, $\gamma(t)$ is the quantity of the stock portfolio, $t \geq 0$. The pair $(\beta(\cdot), \gamma(\cdot))$ describes the state of the bond-stocks securities portfolio at time t . Each of these pairs is called a strategy.

The process $\tilde{X}(t) \triangleq e^{-rt}X(t)$ is said to be the discounted wealth, and the process $\tilde{S}(t) \triangleq e^{-rt}S(t)$ is said to be the discounted stock price.

Definition 3.1 A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be an admissible strategy if $\beta(t)$ and $\gamma(t)$ are random processes which are progressively measurable with respect to the filtration \mathcal{F}_t and such that there exists a sequence of Markov times $\{T_k\}_{k=1}^{+\infty}$ with respect to the filtration \mathcal{F}_t such that $T_k \rightarrow T - 0$ a.s. and

$$\mathbf{E} \int_0^{T_k} (\beta(t)^2 B(t)^2 + S(t)^2 \gamma(t)^2) dt < +\infty \quad \forall k = 1, 2, \dots$$

Definition 3.2 A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be an admissible self-financing strategy, if

$$dX(t) = \beta(t)dB(t) + \gamma(t)dS(t). \quad (3.4)$$

It is well known that (3.4) is equivalent to

$$d\tilde{X}(t) = \gamma(t)d\tilde{S}(t). \quad (3.5)$$

It follows that $\tilde{X}(t)$ is a martingale with respect to the probability measure \mathbf{P}_* .

Let $X(0)$ be an initial wealth, and let $\tilde{X}(t)$ be the discounted wealth generated by an admissible self-financing strategy $(\beta(\cdot), \gamma(\cdot))$. For any Markov time τ such that $\tau \in [0, T]$, we have

$$\mathbf{E}_* \tilde{X}(\tau) = X(0) + \mathbf{E}_* \int_0^\tau \gamma(t)d\tilde{S}(t) = X(0) + \mathbf{E}_* \int_0^\tau \gamma(t)\tilde{S}(t)^{-1}dw_*(t) = X(0).$$

4 The fair price of a general controlled option

Let us consider a controlled option (2.1) with the payoff $F_u = \Phi(u(\cdot), S(\cdot))$, where $\Phi : \mathcal{U} \times C(0, T)$ is a measurable mapping such that $\sup_{u \in \mathcal{U}} \mathbf{E}_* |F_u| < +\infty$. Here \mathcal{U} is the set of all admissible controls $u(\cdot)$. All examples considered above are covered by this general setting.

Definition 4.1 The fair price of an option is the price c such that

- The option writer cannot fulfill option obligations at terminal time T using the wealth raised from the initial wealth $X(0) < c$ with self-financing strategies.
- A rational option buyer wouldn't buy an option for a higher price than c .

The following theorem is formulated for the case of constant r . However, this theorem holds for any model where the risk-neutral measure \mathbf{P}_* exists and is unique; the extension on the case of time variable $r = r(t)$ is straightforward.

Theorem 4.1 *The fair price c_F of an option with the payoff F_u is*

$$c_F = e^{-rT} \sup_{u \in \mathcal{U}} \mathbf{E}_* F_u.$$

Proofs are given in the Appendix.

5 Pricing of options with adapted weight

Consider an option with payoff

$$F_u = g \left(\int_0^T u(t) f(S(t), t) dt \right), \quad (5.1)$$

where $f(x, t) : (0, +\infty) \times [0, T] \rightarrow \mathbf{R}$ and $g(x) : (0, +\infty) \rightarrow \mathbf{R}$ are given continuous non-negative functions such that $|f(x, t)| + |g(x)| \leq \text{const} (|x| + 1)$ and $|\partial f(x, t)/\partial x| + |dg(x)/dx| \leq \text{const}$. In addition, we assume that the function $g(x)$ is non-decreasing.

The function $u(t)$ is the control process that is selected by the option holder.

We assume that $S(t)$ and \mathcal{F}_t are such as described in section 3.

Let \mathcal{U} be the class of processes $u(t)$ consisting of the processes that are adapted to the filtration \mathcal{F}_t and such that

$$u(t) \in [d_0, d_1], \quad (5.2)$$

where $0 \leq d_0 < d_1 < +\infty$.

We consider the class \mathcal{U}_1 of admissible processes $u(t)$ consisting of the processes $u \in \mathcal{U}$ such that

$$\int_0^T u(t) dt = 1. \quad (5.3)$$

To ensure that the set of admissible strategies is non-empty, we assume that $d_0 T < 1$.

By Theorem 4.1, the fair price of this option is

$$c_F = e^{-rT} \sup_{u(\cdot) \in \mathcal{U}_1} \mathbf{E}_* F_u. \quad (5.4)$$

Lemma 5.1 *Assume that the function g is concave on $(0, +\infty)$. In this case, an optimal control for problem (5.4) exists in \mathcal{U}_1 .*

5.1 Pricing via dynamic programming

It follows from the definitions that the price c_F for this option can be found via solution of optimal stopping problem

$$\begin{aligned}
& \text{Maximize} && \mathbf{E}_* g(x(\tau)) \quad \text{over} \quad u(\cdot) \in \mathcal{U}, \\
& \text{subject to} && dx(t) = u(t)f(S(t), t)dt, \\
& && dy(t) = u(t)dt, \\
& && dS(t) = rS(t)dt + \sigma S(t)dw_*(t),
\end{aligned} \tag{5.5}$$

where $\tau = T \wedge \inf\{t \in [0, T] : y(t) \geq 1\}$. In this case, $c_F = e^{-rT} \sup_{u(\cdot) \in \mathcal{U}} \mathbf{E}_* g(x(\tau))$ given that $x(0) = 0$, $y(0) = 0$, $S(0) = S_0$.

Alternatively, the price c_F for this option can be found via solution of optimal stopping stochastic control problem

$$\begin{aligned}
& \text{Maximize} && \mathbf{E}_* g(x(T)) \quad \text{over} \quad u(\cdot) \in \mathcal{U}, \\
& \text{subject to} && dx(t) = \mathbb{I}_{\{y(t) < 1\}} u(t)f(S(t), t)dt, \\
& && dy(t) = u(t)dt, \\
& && dS(t) = rS(t)dt + \sigma S(t)dw_*(t).
\end{aligned} \tag{5.6}$$

In this case, $c_F = e^{-rT} \sup_{u(\cdot) \in \mathcal{U}} \mathbf{E}_* g(x(T))$ given that $x(0) = 0$, $y(0) = 0$, $S(0) = S_0$.

Problem (5.5) and (5.6) are such that the matrix of the diffusion coefficients for the state process is degenerate. In addition, problem (5.5) involves first exit from a domain with a boundary, This makes it difficult to use classical methods of solution. Hence it will be more convenient to use (5.6) that does not feature a boundary and first exit time.

The state equation for problem (5.6) has discontinuous drift coefficient for $x(t)$. To remove this feature, we approximate the problem as the following.

Let functions $\phi_\varepsilon(x, t) : (0, +\infty) \rightarrow \mathbf{R}$ be such as described in Section 6, $\varepsilon > 0$. Let functions $\mathbf{g}_\varepsilon(x) : \mathbf{R} \rightarrow \mathbf{R}$ and $\xi_\varepsilon(y) : \mathbf{R} \rightarrow [0, 1]$ be selected such that the following holds.

- (i) The functions ξ_ε are non-increasing continuously differentiable and such that $\xi_\varepsilon(y) = 1$ for $y < T - \varepsilon$, and $\xi_\varepsilon(y) = 0$ for $y > 1 - \varepsilon + \varepsilon^2$.

- (ii) The functions $\mathbf{g}_\varepsilon(x)$ are bounded and twice differentiable. The corresponding derivatives are bounded, and

$$\mathbf{g}_\varepsilon(x) \rightarrow g(x) \quad \text{as } \varepsilon \rightarrow 0, \quad \mathbf{g}_\varepsilon(x) \leq g(x) \quad \text{for all } x.$$

Let $\mathbf{f}_\varepsilon(u, x, y, s) = u\xi_\varepsilon(y)\phi_\varepsilon(x, t)$.

Consider the stochastic control following problem:

$$\begin{aligned} & \text{Maximize} && \mathbf{E}_* \mathbf{g}_\varepsilon(x(T)) \quad \text{over } u(\cdot) \in \mathcal{U}, \\ & \text{subject to} && dx(t) = \mathbf{f}_\varepsilon(y(t), u(t), S(t), t)dt, \\ & && dy(t) = u(t)dt, \\ & && dS(t) = rS(t)dt + \sigma S(t)dw_*(t). \end{aligned} \tag{5.7}$$

Consider the corresponding value function

$$J_\varepsilon(x, y, s, t) \triangleq \sup_{u(\cdot) \in \mathcal{U}} \mathbf{E}_* \left\{ \mathbf{g}_\varepsilon(x_\varepsilon(T)) \mid x(t) = x, y(t) = y, S(t) = s \right\}. \tag{5.8}$$

Let $D = \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times [0, T]$. Let \mathcal{X} be the class of functions $v(x, y, z, t) : D \rightarrow \mathbf{R}$ such that v is continuous and there exists $c > 0$ such that $v(x, y, z, t) \leq c(|x| + |y| + |z| + 1)$ for all $(x, y, z, t) \in D$. Let \mathcal{X}_1 be the class of functions $v \in \mathcal{X}$ such that v'_x, v'_y , and v'_z belong to \mathcal{X} . Let \mathcal{X}_2 be the class of functions $v \in \mathcal{X}_1$ such that v'_t and v''_{zz} belong to \mathcal{X} .

Theorem 5.1 (i) *The option price can be found as*

$$c_F = \lim_{\varepsilon \rightarrow 0} e^{-rT} J_\varepsilon(0, 0, S(0), 0). \tag{5.9}$$

(ii) *The value function $J = J_\varepsilon$ satisfies the Bellman equation*

$$\begin{aligned} & J_t + \max_{u \in [d_0, d_1]} \{J'_x \mathbf{f}_\varepsilon + J'_y u\} + J'_s r s + \frac{1}{2} J''_{ss} \sigma^2 s^2 = 0, \\ & J(x, y, s, T) = \mathbf{g}_\varepsilon(x). \end{aligned} \tag{5.10}$$

The Bellman equation has unique solution in the class of functions $J = J_\varepsilon(x, y, s, t)$ such that $J_\varepsilon(x, y, s, t) = V_\varepsilon(x, y, \log s, t)$ for some function $V_\varepsilon \in \mathcal{X}_2$. The Bellman equation holds as an equality that is satisfied for a.e. $(x, y, s, t) \in \mathbf{R} \times \mathbf{R} \times (0, +\infty) \times [0, T]$.

5.2 Case of linear g

The dimension of the Bellman equation can be reduced for the case when $g(x) \equiv x$. In this case, the option price c_F can be found via solution of optimal stopping problem

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E}_* \int_0^\tau u(t) f(S(t), t) dt, \quad \text{over } u(\cdot) \in \mathcal{U}, \\ \text{subject to} \quad & dy(t) = u(t) dt, \\ & dS(t) = rS(t) dt + \sigma S(t) dw_*(t), \end{aligned} \quad (5.11)$$

where $\tau = T \wedge \inf\{t \in [0, T] : y(t) \geq 1\}$. In this case, $c_F = e^{-rT} \sup_{u(\cdot) \in \mathcal{U}} \mathbf{E}_* \int_0^\tau u(t) f(S(t), t) dt$ given that $y(0) = 0$, $S(0) = S_0$. Consider the corresponding value function

$$\bar{J}(y, s, t) \triangleq \sup_{u(\cdot) \in \mathcal{U}} \mathbf{E}_* \left\{ \int_t^{\tau_u^{x,t}} u(s) f(S(s), s) ds \mid y(t) = y, S(t) = s \right\}. \quad (5.12)$$

Here

$$\tau_u^{y,s} = T \wedge \inf\{\theta \in [t, T] : y + \int_s^\theta u(q) dq \geq 1\}.$$

The option price is $c_F = e^{-rT} \bar{J}(0, S(0), 0)$. The Bellman equation satisfied formally by \bar{J} is

$$\begin{aligned} \bar{J}_t + \max_{u \in [d_0, d_1]} \{ \bar{J}'_y u + u f(s, t) \} + \bar{J}'_s r s + \frac{1}{2} \bar{J}''_{ss} \sigma^2 s^2 &= 0, \\ \bar{J}(1, s, T) &= 0, \quad \bar{J}(y, s, T) = 0. \end{aligned} \quad (5.13)$$

The Bellman equation holds for $x > 0$, $y < 1$, $s > 0$, $t < T$. However, to derive this equation and prove Verification Theorem, one have to overcome again some technical difficulties arising from the presence of boundary and from the fact that the diffusion in the state equation is degenerate. Instead, we suggest to use an alternative stochastic control problem

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E}_* \int_0^T \mathbb{I}_{\{y(t) \leq 1\}} u(t) f(S(t), t) dt, \quad \text{over } u(\cdot) \in \mathcal{U}, \\ \text{subject to} \quad & dy(t) = u(t) dt, \\ & dS(t) = rS(t) dt + \sigma S(t) dw_*(t). \end{aligned}$$

This problem without does not involve first exit time. The Bellman equation satisfied formally by its value function $J = J(y, s, t)$ is

$$\begin{aligned} J_t + \max_{u \in [d_0, d_1]} \{ J'_x u f + J'_y u + \mathbb{I}_{\{y \leq 1\}} u f \} + J'_s r s + \frac{1}{2} J''_{ss} \sigma^2 s^2 &= 0, \\ J(y, s, T) &= 0. \end{aligned}$$

The question is degenerate again, so we will use an equation with more regular coefficients as an approximation.

Let functions \mathbf{f}_ε be such as described above. Consider stochastic control problem

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E}_* \int_0^T \mathbf{f}_\varepsilon(u(t), y(t), S(t), t) dt, \quad \text{over } u(\cdot) \in \mathcal{U}, \\ \text{subject to} \quad & dy(t) = u(t) dt, \\ & dS(t) = rS(t) dt + \sigma S(t) dw_*(t). \end{aligned} \quad (5.14)$$

Consider the corresponding value function

$$J_\varepsilon(y, s, t) \triangleq \sup_{u(\cdot) \in \mathcal{U}} \mathbf{E}_* \left\{ \int_t^T \mathbf{f}_\varepsilon(u(t), y(t), S(t)) dt \mid y(t) = y, S(t) = s \right\}. \quad (5.15)$$

Let $D' = \mathbf{R} \times \mathbf{R} \times [0, T]$. Let \mathcal{Y} be the class of functions $v(y, z, t) : D' \rightarrow \mathbf{R}$ such that v is continuous and there exists $c > 0$ such that $v(y, z, t) \leq c(|y| + |z| + 1)$ for all $(y, z, t) \in D'$. Let \mathcal{Y}_1 be the class of functions $v \in \mathcal{Y}$ such v'_y and v'_z belong to \mathcal{Y} . Let \mathcal{Y}_2 be the class of functions $v \in \mathcal{Y}_1$ such v'_t and v''_{zz} both belong to \mathcal{Y} .

Theorem 5.2 (i) *The option price can be found as*

$$c_F = \lim_{\varepsilon \rightarrow 0} e^{-rT} J_\varepsilon(0, S(0), 0). \quad (5.16)$$

(ii) *The value function $J = J_\varepsilon(y, s, t)$ for problem (5.14) satisfies the Bellman equation*

$$\begin{aligned} J_t + \max_{u \in [d_0, d_1]} \{J'_y u + \mathbf{f}_\varepsilon\} + J'_s r s + \frac{1}{2} J''_{ss} \sigma^2 s^2 &= 0, \\ J(y, s, T) &= 0. \end{aligned} \quad (5.17)$$

The Bellman equation has unique solution in the class of functions $J = J_\varepsilon(x, y, s, t)$ such that $J_\varepsilon(y, s, t) = V_\varepsilon(y, \log s, t)$ for some function $V_\varepsilon \in \mathcal{Y}_2$. The Bellman equation holds as an equality that is satisfied for a.e. $(y, s, t) \in \mathbf{R} \times (0, +\infty) \times [0, T]$.

5.3 Analog of Merton Theorem

In this section, we consider again a risky asset with the price $S(t)$, where $t \in [0, T]$. Consider an option with the payoff at time T

$$F_u = \int_0^T u(t) f(S(t), t) dt,$$

where $f : (0, +\infty) \times [0, T] \rightarrow \mathbf{R}$ is a given function such that $|f(x, t)| \leq \text{const}(1 + |x|)$ and $f(x, t) \geq 0$. Here $u(t)$ is the control process that is selected by the option holder. The set \mathcal{U}_1 of admissible processes $u(t)$ consists of the processes that are adapted to the current information flow (or to the filtration, generated by $S(t)$ and such that

$$u(t) \in [0, L], \quad \int_0^T u(t)dt \leq 1,$$

where $L \in (0, +\infty)$ is given.

If $TL \leq 1$ then the optimal solution is $u \equiv L$. Hence we assume that $T > L^{-1}$.

This option represents the limit version of multi-exercise options when the distribution of exercise times approaching a continuous distribution. This model can be used, for instance, for energy trading with $u(t)$ representing the quantity of energy purchased at time t for the fixed price K when the market price is above K . The total amount $\int_0^T u(t)dt$ of energy that can be purchased is limited per option.

Merton theorem states that American and European options with the same parameters have the same price and that early exercise is not rational. The following theorem represents a extension of this theorem on the case of controlled options.

Theorem 5.3 *Let $f(S(t), t) = e^{r(T-t)}h(S(t))$, where the function $h(x)$ is convex and non-linear in $x > 0$, and such that at least one of the following conditions holds:*

- (i) *the function $\alpha^{-1}h(\alpha x)$ is non-decreasing in $\alpha \in (0, 1]$; or*
- (ii) *$r = 0$.*

Then $\sup_{u(\cdot) \in \mathcal{U}_1} \mathbf{E}_ F_u$ is achieved for the control process*

$$\hat{u}(t) = \begin{cases} L, & t \geq T - 1/L \\ 0, & t < T - 1/L, \end{cases}$$

and the price of the option is

$$e^{-rT} \mathbf{E}_* \int_0^T \hat{u}(t) f(S(t), t) dt = \frac{e^{-rT}}{L} \mathbf{E}_* \int_{T-1/L}^T f(S(t), t) dt.$$

Remark 5.1 The function $h(x) = (x - K)^+$ is such that assumption (i) of Theorem 5.3 are satisfied (this function corresponds to the call option with continuously distributed payoff time).

However, assumptions (i) of Theorem 5.3 are not satisfied for $h(x) = (K - x)^+$ that corresponds to the put option. The pricing for this case with $r > 0$ is an interesting problem. A solution could be a useful approximation of the classical optimal stopping pricing rule for American option: for the controlled option with restriction that $u(t) \in [0, L]$, the limit case when $L \rightarrow +\infty$ will lead to a Stefan problem and optimal stopping. The approximate solution for finite L may be easier to find since it does not require optimal stopping and solution of Stefan problem. We leave it for future research.

6 Pricing for non-adapted normalized weight $v(t)$

Consider an option with payoff (2.5), where functions $f(x, t)$ and $g(x)$ have the same properties as in Section 5.

The function $u(t)$ is the control process that is selected by the option holder. The set of admissible processes \mathcal{U} is the set of \mathcal{F}_t -adapted processes $u(t)$ that take values in $[d_0, d_1]$, where $0 \leq d_0 < d_1 < +\infty$.

We don't exclude the case when $d_0 = 0$ and $u(t)|_{t \in [0, T)} = 0$. In this case, the payoff is assumed to be

$$F_u = g(f(S(T), T)), \quad (6.1)$$

i.e., as the limit as $\varepsilon \rightarrow 0$ of the payoffs (2.5) defined for $u_\varepsilon(t) \equiv d_1 \mathbb{I}_{\{t > T - \varepsilon\}}$.

For $\varepsilon > 0$, let functions $f_\varepsilon(u, x, t) : (0, +\infty) \rightarrow \mathbf{R}$ and $g_\varepsilon(x, y) : \mathbf{R}^2 \rightarrow \mathbf{R}$ be selected such that the following holds.

- (i) $f_\varepsilon(u, x, t) = h_\varepsilon(u, t)\phi_\varepsilon(x, t)$, where $h_\varepsilon : [d_0, d_1] \times [0, T] \rightarrow \mathbf{R}$ and $\phi_\varepsilon : (0, +\infty) \times [0, T] \rightarrow \mathbf{R}$ are measurable functions with the following properties.

- (a) The functions $\hat{\phi}_\varepsilon(z, t) \triangleq \phi_\varepsilon(e^z, t)$ are bounded and continuously differentiable in $(z, t) \in \mathbf{R} \times (0, T)$. The corresponding derivatives are bounded.

- (b) $h_\varepsilon(u, t) = u(1 - \psi_\varepsilon(t)) + d_1\psi_\varepsilon(t)$, where $\psi_\varepsilon(t) : \mathbf{R} \rightarrow [0, 1]$ is a continuously differentiable non-decreasing function such that $\psi_\varepsilon(t) = 0$ for $t < T - \varepsilon$, $\psi_\varepsilon(t) = d_1$, $t > T - \varepsilon + \varepsilon^2$.

- (ii) $\phi_\varepsilon(x, t) \leq f(x, t)$, $g_\varepsilon(x, y) \leq g(x/y)$ for all $x, y \neq 0, t$.

- (iii) The functions $g_\varepsilon(x, y)$ are bounded and twice differentiable in (x, y) . The corresponding derivatives are bounded.
- (iv) (a) $\phi_\varepsilon(x, t) \rightarrow f(x, t)$ as $\varepsilon \rightarrow 0$ for all x, t .
 (b) If $y \neq 0$ then $g_\varepsilon(x, y) \rightarrow g(x/y)$ as $\varepsilon \rightarrow 0$ for all x .
 (c) If, for some $c \in \mathbf{R}$, we have that $\varepsilon \rightarrow 0$, $y \rightarrow 0$, $x/y \rightarrow c$, then $g_\varepsilon(x, y) \rightarrow g(c)$.

Consider optimal stochastic control problem

$$\begin{aligned}
 &\text{Maximize} && \mathbf{E}_* g_\varepsilon(x(T), y(T)) \quad \text{over} \quad u(\cdot) \in \mathcal{U}, \\
 &\text{subject to} && dx(t) = f_\varepsilon(u(t), S(t), t)dt, \\
 &&& dy(t) = h_\varepsilon(u(t), t)dt, \\
 &&& dS(t) = rS(t)dt + \sigma S(t)dw_*(t).
 \end{aligned} \tag{6.2}$$

For $u \in \mathcal{U}$, set

$$\mathcal{J}_\varepsilon(u, x, y, s, t) \triangleq \mathbf{E}_* \left\{ g_\varepsilon(x(T), y(T)) \mid x(t) = x, y(t) = y, S(t) = s \right\}.$$

Consider the corresponding value function

$$J_\varepsilon(x, y, s, t) \triangleq \sup_{u(\cdot) \in \mathcal{U}} \mathcal{J}_\varepsilon(u, x, y, s, t). \tag{6.3}$$

Let \mathcal{X}_2 be the space introduced in Section 5.

Theorem 6.1 (i) *The option price can be found as*

$$c_F = \lim_{\varepsilon \rightarrow 0} e^{-rT} J_\varepsilon(0, 0, S(0), 0). \tag{6.4}$$

(ii) *The value function $J = J_\varepsilon$ satisfies the Bellman equation*

$$\begin{aligned}
 &J_t + \max_{u \in [d_0, d_1]} \{J'_x f_\varepsilon + J'_y h_\varepsilon\} + J'_s r s + \frac{1}{2} J''_{ss} \sigma^2 s^2 = 0, \\
 &J(x, y, s, T) = g_\varepsilon(x, y).
 \end{aligned} \tag{6.5}$$

The Bellman equation has unique solution in the class of functions $J = J_\varepsilon(x, y, s, t)$ such that $J_\varepsilon(x, y, s, t) = V_\varepsilon(x, y, \log s, t)$ for some function $V_\varepsilon \in \mathcal{X}_2$. The Bellman equation holds as an equality that is satisfied for a.e. $(x, y, s, t) \in \mathbf{R} \times \mathbf{R} \times (0, +\infty) \times [0, T]$.

Appendix: Proofs

Proof of Theorem 4.1. It is known that the discounted wealth is a martingale under \mathbf{P}_* for any admissible strategies. For a given $u(\cdot)$, the ability to fulfill the option obligations means that $X(T) \geq F_u$ for any $u(\cdot)$ i.e., $\tilde{X}(T) \geq e^{-rT} F_u$. Hence $X(0) = \mathbf{E}_* \tilde{X}(T) \geq e^{-rT} F_u$. It follows that

$$c_F \geq e^{-rT} \sup_{u \in \mathcal{U}} \mathbf{E}_* F_u.$$

Further, suppose that there exists $\varepsilon > 0$ such that, for all $u(\cdot) \in \mathcal{U}$,

$$c_F \geq e^{-rT} \mathbf{E}_* F_u + \varepsilon.$$

In this case, for any strategy $u(\cdot) \in \mathcal{U}$, the claim F_u can be replicated with the initial wealth $X_0 \leq c_F - \varepsilon$. Therefore, any potential option buyer could save $\varepsilon > 0$ quantity of cash if she select to replicate the payoff F_u with some self-financing strategy. Therefore, a rational option buyer would't buy an option for the price c_F . This completes the proof. \square

Proof of Lemma 5.1. Let \mathcal{H} be the Hilbert space formed as the completion of the set of all square integrable and adapted processes in the norm of $L_2([0, T] \times \Omega)$. The set \mathcal{U}_1 is a convex and closed (and, therefore, weakly closed) subset of \mathcal{H} . Hence \mathcal{U}_1 is compact in the weak topology of \mathcal{H} . Consider the mapping $\phi : \mathcal{U}_1 \rightarrow \mathbf{R}$ such that $\phi(u) = \mathbf{E}_* F_u$. Let $\{u_j\} \subset \mathcal{H}$ be a sequence such that

$$\phi(u_j) \rightarrow \sup_{u \in \mathcal{H}} \phi(u) \quad \text{as } j \rightarrow +\infty. \quad (\text{A.1})$$

There exists a subsequence $\{u_k\}$ and $\bar{u} \in \mathcal{U}_1$ such that $u_k \rightarrow \bar{u} \in \mathcal{H}$ weakly in \mathcal{H} as $k \rightarrow +\infty$. By Mazur's Theorem (Theorem 5.1.2 from Yosida (1995)), there exists a sequence of integer numbers $k = k_i \rightarrow +\infty$ such that there exist sets of real numbers $\{a_{mk}\}_{m=1}^k \subset [0, 1]$ such that $\sum_{m=1}^k a_{mk} = 1$ and that

$$\tilde{u}_k \triangleq \sum_{m=1}^k a_{mk} u_m \rightarrow \bar{u} \quad \text{in } \mathcal{H}. \quad (\text{A.2})$$

In addition, there exists a subsequence $\{\hat{u}_m\}$ of this sequence such that $\hat{u}_m \rightarrow \bar{u}$ a.e. as $k \rightarrow +\infty$. Consider mappings $G(u, y) : \mathcal{U}_1 \times C([0, T] \rightarrow \mathbf{R}) \rightarrow \mathbf{R}$ and $I(u, y) : \mathcal{U}_1 \times C([0, T] \rightarrow \mathbf{R}) \rightarrow \mathbf{R}$ such that $I(u, y) = \int_0^T u(t) f(y(t), t) dt$ and $G(u, y) = g(I(u, y))$. Clearly, $G(\hat{u}_k, S(\cdot)) \rightarrow G(\bar{u}, S(\cdot))$ a.s. as

$k \rightarrow +\infty$. In addition,

$$\begin{aligned} |I(u, S)| &\leq T \max_{t \in [0, T]} |f(S(t), t)| \max u(t) \leq \text{const} \cdot d_1 T \max_{t \in [0, T]} (S(t) + 1), \\ |G(u, S)| &\leq \text{const} I(u, S(\cdot)). \end{aligned}$$

By the Lebesgue's Dominated Convergence Theorem, it follows that $\phi(\hat{u}_k) \rightarrow \phi(\bar{u})$.

By linearity of I , we have that $I(\hat{u}_k, S(\cdot)) = \sum_{m=1}^k a_{mk} I(u_m, S(\cdot))$. By the concavity of g , it follows that

$$g(I(\hat{u}_k, S(\cdot))) \geq g\left(\sum_{m=1}^k a_{mk} I(u_m, S(\cdot))\right).$$

Hence

$$\phi(\hat{u}_k) \geq \sum_{m=1}^k a_{mk} \phi(u_m) \rightarrow \sup_{u \in \mathcal{U}_1} \phi(u) \quad \text{as } k \rightarrow +\infty.$$

Hence $\phi(\bar{u}) = \sup_{u \in \mathcal{H}} \phi(u)$, i.e., \bar{u} is a optimal control. \square

Proof of Theorem 5.1. Let us prove statement (i). By Lemma 5.1, $c_F = e^{-rT} \mathbf{E}_* F_{\bar{u}} g$ for some $\hat{u} \in \mathcal{U}_1$. Let $\hat{y}(t) = \int_0^t \hat{u}(s) ds$. By the assumptions on $\mathbf{g}_\varepsilon, \phi_\varepsilon$, it follows that $e^{-rT} J_\varepsilon(0, 0, S(0), 0) \leq c_F$ for any $\varepsilon > 0$ and

$$J_\varepsilon(0, 0, S(0), 0) \geq \mathbf{E}_* \mathbf{g}_\varepsilon \left(\int_0^T \mathbf{f}_\varepsilon(\hat{u}(t), \hat{y}(t), S(t), t) dt \right).$$

Moreover,

$$\mathbf{g}_\varepsilon \left(\int_0^T \mathbf{f}_\varepsilon(\hat{u}(t), \hat{y}(t), S(t), t) dt \right) \rightarrow g \left(\int_0^T \hat{u}(t) \mathbb{I}_{\{\hat{y}(t) < 1\}} f(S(t), t) dt \right) = F_{\bar{u}} \quad \text{a.s. as } \varepsilon \rightarrow 0.$$

By the Lebesgue's Dominated Convergence Theorem and by the assumptions on $\mathbf{g}_\varepsilon, \phi_\varepsilon$, statement (i) follows.

Let us prove statement (ii). Let us consider the change of variables $R(t) = \ln S(t)$. Using the Ito formula, we obtain that this change of variables transfers the corresponding control problem as

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E}_* \mathbf{g}_\varepsilon(x(T)) \quad \text{over } u(\cdot) \in \mathcal{U}, \\ \text{subject to} \quad & dx(t) = \mathbf{f}_\varepsilon(u(t), y(t), e^{R(t)}, t) dt, \\ & dy(t) = u(t) dt, \\ & dR(t) = (r - \sigma^2/2) dt + \sigma dw_*(t). \end{aligned}$$

Consider the corresponding value function

$$V(x, y, z, t) \triangleq \sup_{u(\cdot) \in \mathcal{U}} \mathbf{E}_* \left\{ \mathbf{g}_\varepsilon(x(T)) \mid x(t) = x, y(t) = y, R(t) = z \right\}.$$

Again, the coefficients of this problem are such that the assumptions of Theorem 4.1.4 and Theorem 4.4.3 from Krylov (1980), p.167,192, are satisfied.

By Theorem 4.1.4 from Krylov (1980), p.167, this function satisfy the corresponding parabolic Bellman equation that has unique solution $V \in \mathcal{X}_1$. The Bellman equation holds in the generalized sense, i.e., as an equality of the distributions. This equation includes only one partial derivative of the second order, V''_{zz} presented with the coefficient $\sigma^2/2 > 0$. By Theorem 4.4.3 from Krylov (1980), p.192, the derivative $V'_t(x, y, z, t)$ belongs to \mathcal{X} . It follows that $V''_{zz} \in \mathcal{X}$. The Bellman equation for $J(x, y, s, t) = V(x, y, \log s, t)$ is defined by the equation for V with the corresponding change of variables. Then the proof of statement (ii) and Theorem 5.1 follows. \square .

Proof of Theorem 5.2 is similar to the proof of Theorem 5.1.

Proof of Theorem 5.3. First, note that, by Lemma 5.1, it follows that a optimal control $u \in \mathcal{U}_1$ exists. Consider a linear mapping $\phi(u) : \mathcal{U}_1 \rightarrow \mathbf{R}$ such that $\phi(u) = \mathbf{E}_* F_u$.

Let $u(\cdot) \neq \hat{u}(\cdot)$ be a process in \mathcal{U}_1 . Let us show that $u(\cdot)$ cannot be optimal. Since $u(\cdot) \neq \hat{u}(\cdot)$, there exist a equivalent in $L_2([0, T] \times \Omega)$ version of $u(\cdot)$, $v \in [0, L]$, non-random times $t_0 \in [0, T - L^{-1}]$, $t_1 \in (T - L^{-1}, T]$ and a set $\Omega_0 \in \mathcal{F}_{t_0}$, such that $\mathbf{P}_*(\Omega_0) > 0$, t_k are Lebesgue points for $u(t, \omega)f(S(t, \omega), t)$ for all $\omega \in \Omega_0$, and, for small enough $\varepsilon > 0$,

$$u(t_0, \omega) > v, \quad V(t, \omega) \triangleq u(t, \omega) + u(t - t_1 + t_0, \omega) - v \in [0, L] \quad \forall t \in J_\varepsilon, \omega \in \Omega_0,$$

where $J_\varepsilon \triangleq [t_1 - \varepsilon, t_1]$. Note that feasibility of the property that $V(t, \omega) \in [0, L]$ can be seen from the existence of t_0, t_1, Ω_0 such that $\text{ess sup}_{t \in I_\varepsilon} (u(t, \omega) - v) < 0$ and $\text{ess sup}_{t \in J_\varepsilon} u(t, \omega) < L$ for all $\omega \in \Omega_0$.

Let $I_\varepsilon \triangleq [t_0, t_0 + \varepsilon]$ and let $u_\varepsilon(t)$ be constructed as the following:

$$u_\varepsilon(t) = \begin{cases} u(t), & t \notin I_\varepsilon \cup J_\varepsilon, \\ v, & t \in I_\varepsilon, \\ V(t), & t \in J_\varepsilon. \end{cases}$$

Let us show that $u_\varepsilon(\cdot) \in \mathcal{U}_1$ for small enough $\varepsilon > 0$. Clearly, this process is adapted to \mathcal{F}_t . By the definition of \mathcal{U}_1 ,

$$\int_0^T u(t)dt = \int_0^{t_0} u(t)dt + \int_{t_1}^T u(t)dt + \int_{I_\varepsilon \cup J_\varepsilon} u(t)dt = 1$$

and

$$\begin{aligned} \int_0^T u_\varepsilon(t)dt &= \int_0^{t_0} u(t)dt + \int_{t_1}^T u(t)dt + \varepsilon v + \int_{J_\varepsilon} V(t)dt \\ &= \int_0^{t_0} u(t)dt + \int_{t_1}^T u(t)dt + \varepsilon v + \int_{J_\varepsilon} u(t)dt + \int_{J_\varepsilon} u(t - (t_1 - t_0))dt - \varepsilon v \\ &= \int_0^T u(t)dt = 1, \end{aligned}$$

since $\int_{J_\varepsilon} u(t - t_1 + t_0)dt = \int_{J_\varepsilon} u(t - t_1 + t_0)dt = 1$. Hence $u_\varepsilon(\cdot) \in \mathcal{U}_1$ for small enough $\varepsilon > 0$.

To prove that $u(\cdot)$ is not optimal, it suffices to show that

$$\lim_{\varepsilon \rightarrow 0+} \frac{\phi(u_\varepsilon) - \phi(u)}{\varepsilon} > 0. \quad (\text{A.3})$$

We have that

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0+} \frac{\phi(u_\varepsilon) - \phi(u)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0+} \varepsilon^{-1} \mathbf{E}_* \mathbb{I}_{\Omega_0} \left[\int_{I_\varepsilon} (v - u(t))f(S(t), t)dt + \int_{J_\varepsilon} (V(t) - u(t))f(S(t), t)dt \right] \\ &= \lim_{\varepsilon \rightarrow 0+} \mathbf{E}_* \mathbb{I}_{\Omega_0} \left[\varepsilon^{-1} \int_{I_\varepsilon} (v - u(t))f(S(t), t)dt + \varepsilon^{-1} \int_{J_\varepsilon} (V(t) - u(t))f(S(t), t)dt \right] \\ &= \mathbf{E}_* \mathbb{I}_{\Omega_0} [(v - u(t_0))f(S(t_0), t_0) + (u(t_1) + u(t_0) - v - u(t_1))f(S(t_1), t_1)] \\ &= \mathbf{E}_* \mathbb{I}_{\Omega_0} [(v - u(t_0))f(S(t_0), t_0) + (u(t_0) - v)f(S(t_1), t_1)] \\ &= \mathbf{E}_* \mathbb{I}_{\Omega_0} \mathbf{E}_* \{ (v - u(t_0))f(S(t_0), t_0) + (u(t_0) - v)f(S(t_1), t_1) | \mathcal{F}_{t_0} \} \\ &= \mathbf{E}_* \mathbb{I}_{\Omega_0} (u(t_0) - v) [\mathbf{E}_* \{ f(S(t_1), t_1) | \mathcal{F}_{t_0} \} - f(S(t_0), t_0)]. \end{aligned} \quad (\text{A.4})$$

Further, we have that $\tilde{S}(t) \triangleq S(t)e^{-rt}$ is a martingale under \mathbf{P}_* . We have also that the support of the conditional distribution of $S(t_1)$ given $S(t_0)$ is $(0, +\infty)$. Since $g(\cdot)$ is convex and non-linear, it follows from the Jensen's inequality that

$$\mathbf{E}_* \{ h(S(t_1)) | \mathcal{F}_{t_0} \} = \mathbf{E}_* \{ h(e^{rt_1} \tilde{S}(t_1)) | \mathcal{F}_{t_0} \} > h(e^{rt_1} \tilde{S}(t_0)) = h(e^{r(t_1 - t_0)} S(t_0)).$$

By the properties of h , we have that

$$e^{-r(t_1 - t_0)} h(e^{r(t_1 - t_0)} S(t_0)) \geq h(S(t_0)).$$

Hence

$$h(e^{r(t_1-t_0)}S(t_0)) \geq e^{r(t_1-t_0)}h(S(t_0)),$$

and

$$\mathbf{E}_*\{h(S(t_1))|F_{t_0}\} > h(e^{r(t_1-t_0)}S(t_0)) \geq e^{r(t_1-t_0)}h(S(t_0)).$$

Hence

$$\mathbf{E}_*\{f(S(t_1), t_1)|F_{t_0}\} = e^{r(T-t_1)}\mathbf{E}_*\{h(S(t_1))|F_{t_0}\} > e^{r(T-t_1)}e^{r(t_1-t_0)}h(S(t_0)) = f(S(t_0), t_0).$$

By (A.4), we obtain that limit (A.4) is positive. Hence (A.3) holds and the proof follows. \square

Proof of Theorem 6.1. Let us prove statement (i). Let $\{u_i\} \subset \mathcal{U}$ be a sequence such that

$$\mathbf{E}_*F_{u_i} \rightarrow \sup_{u \in \mathcal{U}} \mathbf{E}_*F_u \quad \text{as } i \rightarrow +\infty. \quad (\text{A.5})$$

Let F'_u be defined similarly to F_u with (f, g) replaced by $(\phi_\varepsilon, g_\varepsilon)$. Clearly, for any $u \in \mathcal{U}$, there exists $\hat{u} \in \mathcal{U}$ such that $\mathbf{E}_*F'_u = \mathcal{J}_\varepsilon(u, 0, 0, S(0), 0)$. Hence $J_\varepsilon(0, 0, S(0), 0) \leq \sup_{u \in \mathcal{U}} \mathbf{E}_*F'_u$. By the properties of $(f_\varepsilon, g_\varepsilon)$, it follows that $\sup_{u \in \mathcal{U}} \mathbf{E}_*F'_u \leq e^{rT}c_F$. Hence $e^{-rT}J_\varepsilon(0, 0, S(0), 0) \leq c_F$ for any $\varepsilon > 0$. Moreover,

$$J_\varepsilon(0, 0, S(0), 0) \geq \mathbf{E}_*g_\varepsilon \left(\int_0^T f_\varepsilon(u_i(t), S(t), t)dt, \int_0^T h_\varepsilon(u_i(t), t)dt \right).$$

Let i be fixed. If $\int_0^T u_i(t)dt > 0$ then, by the Lebesgue's Dominated Convergence Theorem and by the assumptions on g, f ,

$$g_\varepsilon \left(\int_0^T f_\varepsilon(u_i(t), S(t), t)dt, \int_0^T h_\varepsilon(u_i(t), t)dt \right) \rightarrow g \left(\frac{\int_0^T u_i(t)f(S(T), t)dt}{\int_0^T u_i(t)dt} \right) = F_{u_i} \quad \text{a.s. as } \varepsilon \rightarrow 0.$$

If $\int_0^T u_i(t)dt = 0$ then, by assumption (6.1),

$$\begin{aligned} & g_\varepsilon \left(\int_0^T f_\varepsilon(u_i(t), S(t), t)dt, \int_0^T h_\varepsilon(u_i(t), t)dt \right) \\ &= g_\varepsilon \left(\int_{T-\varepsilon}^{T-\varepsilon+\varepsilon^2} h_\varepsilon(u_i(t), t)f(S(t), t)dt + \int_{T-\varepsilon+\varepsilon^2}^T d_1f(S(t), t)dt, \right. \\ & \quad \left. \int_{T-\varepsilon}^{T-\varepsilon+\varepsilon^2} h_\varepsilon(u_i(t), t)dt + \int_{T-\varepsilon+\varepsilon^2}^T d_1dt \right) \\ &= g_\varepsilon \left(O(\varepsilon^2) + \int_{T-\varepsilon+\varepsilon^2}^T d_1f(S(t), t)dt, O(\varepsilon^2) + (\varepsilon - \varepsilon^2)d_1 \right) \\ & \rightarrow g(f(S(T), T)) = F_{u_i} \quad \text{a.s. as } \varepsilon \rightarrow 0 \end{aligned}$$

again. By the Lebesgue's Dominated Convergence Theorem and by the assumptions on g, f again, it follows that

$$\mathcal{J}_\varepsilon(u_i, 0, 0, S(0), 0) \rightarrow \mathbf{E}_* F_{u_i} \quad \text{as } \varepsilon \rightarrow 0. \quad (\text{A.6})$$

We now in the position to prove statement (i). It suffices to show that, for any $\delta > 0$, there exists $\varepsilon_* > 0$ such that $J_\varepsilon(0, 0, S(0), 0) \geq \sup_{u \in \mathcal{U}} \mathbf{E}_* F_u - \delta$ for $\varepsilon < \varepsilon_*$. Let i be such that $\mathbf{E}_* F_{u_i} \geq \sup_{u \in \mathcal{U}} \mathbf{E}_* F_u - \delta/2$. By (A.6), there exists $\varepsilon_* = e_*(\delta, i) > 0$ such that $\mathcal{J}_\varepsilon(u_i, 0, 0, S(0), 0) \geq \mathbf{E}_* F_{u_i} - \delta/2$ for all $\varepsilon < \varepsilon_1$. Hence $J_\varepsilon(0, 0, S(0), 0) \geq \mathbf{E}_* F_{u_i} - \delta/2$ and $J_\varepsilon(0, 0, S(0), 0) \geq \sup_{u \in \mathcal{U}} \mathbf{E}_* F_u$ for these ε . Then statement (i) follows.

Let us prove statement (ii). Let us consider the change of variables $R(t) = \ln S(t)$. Using the Ito formula, we obtain that this change of variables transfers the corresponding control problem as

$$\text{Maximize} \quad \mathbf{E}_* g_\varepsilon(x(T), y(T)) \quad \text{over } u(\cdot) \in \mathcal{U}, \quad (\text{A.7})$$

$$\begin{aligned} \text{subject to} \quad & dx(t) = f_\varepsilon(u(t), e^{R(t)}, t)dt, \\ & dy(t) = h_\varepsilon(u(t), t)dt, \\ & dR(t) = (r - \sigma^2/2)dt + \sigma dw_*(t). \end{aligned} \quad (\text{A.8})$$

Consider the corresponding value function

$$V(x, y, z, t) \triangleq \sup_{u(\cdot) \in \mathcal{U}} \mathbf{E}_* \left\{ g_\varepsilon(x(T), y(T)) \mid x(t) = x, y(t) = y, R(t) = z \right\}. \quad (\text{A.9})$$

Note that the coefficients of this problem are such that the assumptions of Theorem 4.1.4 and Theorem 4.4.3 from Krylov (1980), p.167,192, are satisfied. The remaining part of the proof repeats the proof of Theorem 5.1(ii). This completes the proof of Theorem 6.1. \square

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